First Derivative Test for Local Extrema

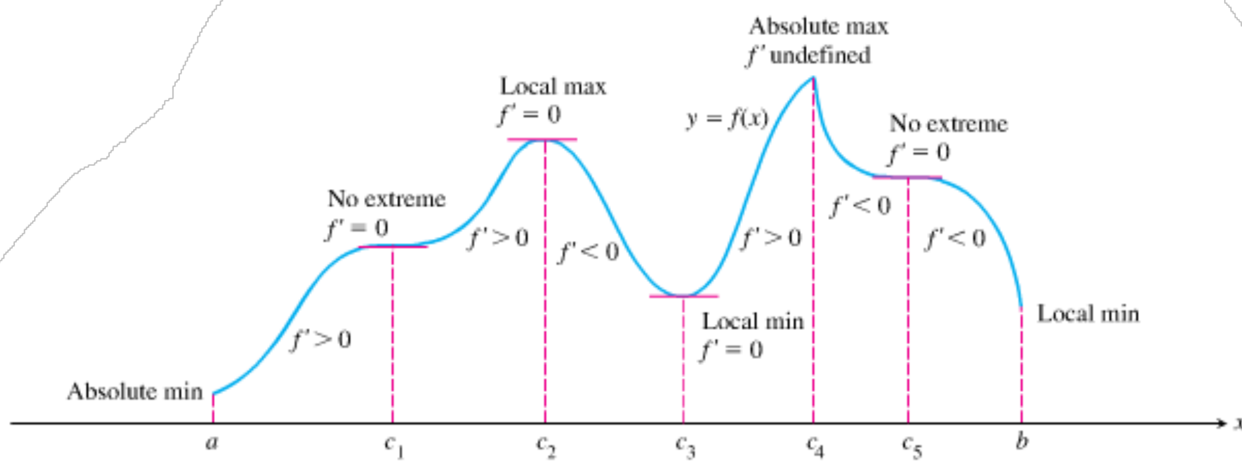


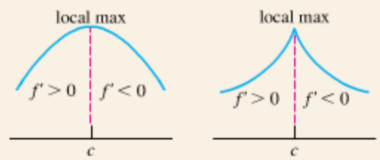
Figure 4.18 A function's first derivative tells how the graph rises and falls.

THEOREM 4 **First Derivative Test for Local Extrema**

The following test applies to a continuous function $f(x)$.

At a critical point c:

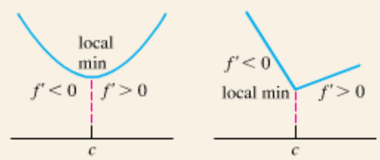
1. If f' changes sign from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

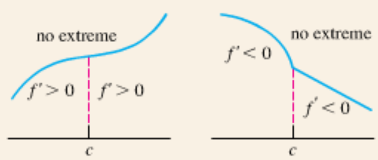
continued

2. If f' changes sign from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

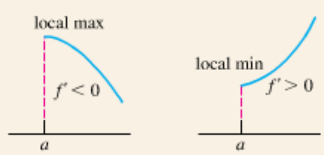
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

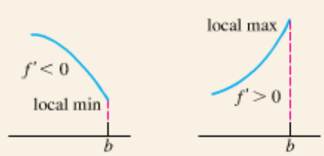
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.



EXAMPLE 1 **Using the First Derivative Test**

For each of the following functions, use the First Derivative Test to find the local extreme values. Identify any absolute extrema.

(a) $f(x) = x^3 - 12x - 5$ **(b)** $g(x) = (x^2 - 3)e^x$

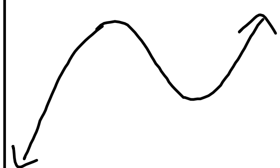
a) $3x^2 - 12$

$3(x^2 - 4)$

$3(x+2)(x-2) = 0$

$+ \quad - \quad +$

$-2 \quad 2$



no absolute extrema

local max at $x = -2$ because $f'(x)$ changes from pos. to neg.

local min at $x = 2$ because $f'(x)$ changes from neg. to pos.

b) $g(x) = (x^2 - 3)e^x$

$g'(x) = (x^2 - 3)e^x + 2xe^x$

$= x^2e^x - 3e^x + 2xe^x$

$= e^x(x^2 + 2x - 3)$

$= e^x(x+3)(x-1)$ $\quad + \quad - \quad +$

$-3 \quad 1$

on left side

$\lim_{x \to -\infty} g(x) = 0$

at $x = 1$

$g(1) \approx -5.437 \rightarrow$ absolute min because it is < 0 and $g'(x)$ changes from neg. to pos.

no absolute max

local max at $x = -3$ because $g'(x)$ changes from pos. to neg.

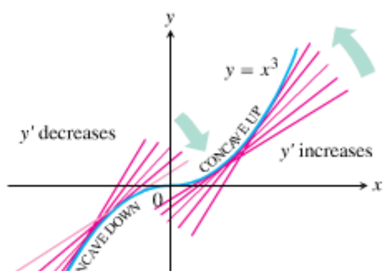
Concavity

DEFINITION Concavity

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if y' is increasing on I.

(b) **concave down** on an open interval I if y' is decreasing on I.



Concavity Test

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$.

(b) concave down on any interval where $y'' < 0$.

EXAMPLE 2 Determining Concavity

Use the Concavity Test to determine the concavity of the given functions on the given intervals:

(a) $y = x^2$ on $(3, 10)$ **(b)** $y = 3 + \sin x$ on $(0, 2\pi)$

Points of Inflection

DEFINITION **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

EXAMPLE 3 Finding Points of Inflection

Find all points of inflection of the graph of $y = e^{-x^2}$.

EXAMPLE 4 Reading the Graph of the Derivative

The graph of the *derivative* of a function f on the interval $[-4, 4]$ is shown in Figure 4.25. Answer the following questions about f, justifying each answer with information obtained from the graph of f'.

(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave up?

(c) At which x-coordinates does f have local extrema?

(d) What are the x-coordinates of all inflection points of the graph of f?

(e) Sketch a possible graph of f on the interval $[-4, 4]$.

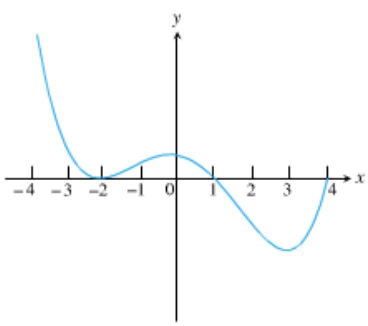
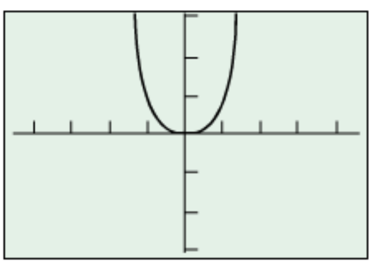


Figure 4.25 The graph of f', the derivative of f, on the interval $[-4, 4]$.

Caution: It is tempting to oversimplify a point of inflection as a point where the second derivative is zero, but that can be wrong for two reasons:

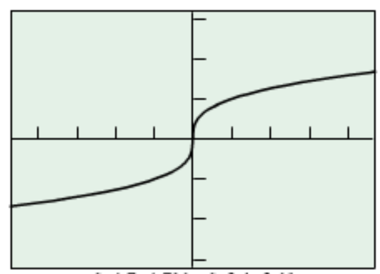
1. *The second derivative can be zero at a noninflection point.* For example, consider the function $f(x) = x^4$ (Figure 4.27). Since $f''(x) = 12x^2$, we have $f''(0) = 0$; however, $(0, 0)$ is not an inflection point. Note that f'' does not *change sign* at $x = 0$.
2. *The second derivative need not be zero at an inflection point.* For example, consider the function $f(x) = \sqrt[3]{x}$ (Figure 4.28). The concavity changes at $x = 0$, but there is a *vertical* tangent line, so both $f'(0)$ and $f''(0)$ fail to exist.

Therefore, the only safe way to test algebraically for a point of inflection is to confirm a sign change of the second derivative. This *could* occur at a point where the second derivative is zero, but it also could occur at a point where the second derivative fails to exist.



[– 4.7, 4.7] by [–3.1, 3.1]

Figure 4.27 The function $f(x) = x^4$ does not have a point of inflection at the origin, even though $f''(0) = 0$.



[–4.7, 4.7] by [–3.1, 3.1]

Figure 4.28 The function $f(x) = \sqrt[3]{x}$ has a point of inflection at the origin, even though $f''(0) \neq 0$.

EXAMPLE 5 Studying Motion along a Line

A particle is moving along the x-axis with position function

$$x(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Second Derivative Test for Local Extrema

THEOREM 5 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

EXAMPLE 7 Using the Second Derivative Test

Find the local extreme values of $f(x) = x^3 - 12x - 5$.

EXAMPLE 8 Using f' and f'' to Graph f

Let $f'(x) = 4x^3 - 12x^2$.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

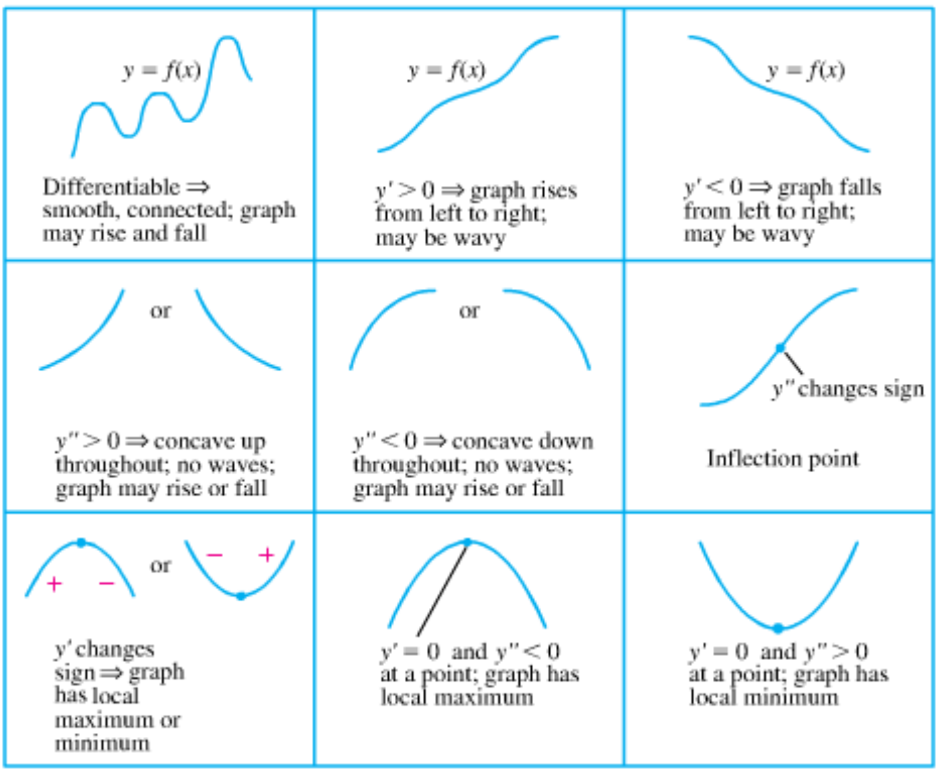
(d) Sketch a possible graph for f.

EXPLORATION 1 Finding f from f'

Let $f'(x) = 4x^3 - 12x^2$.

1. Find three different functions with derivative equal to $f'(x)$. How are the graphs of the three functions related?
2. Compare their behavior with the behavior found in Example 8.

Learning about Functions from Derivatives



EXAMPLE 9 Analyzing a Discontinuous Derivative

A function f is continuous on the interval $[-4, 4]$. The discontinuous function f', with domain $[-4, 0) \cup (0, 2) \cup (2, 4]$, is shown in the graph to the right (Figure 4.33).

(a) Find the x-coordinates of all local extrema and points of inflection of f.

(b) Sketch a possible graph of f.

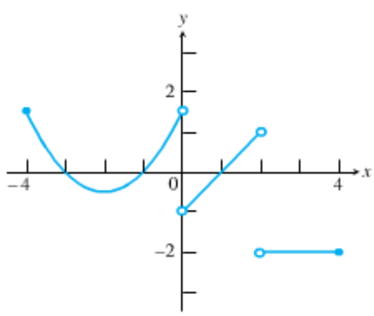


Figure 4.33 The graph of f', a discontinuous derivative.

EXPLORATION 2 **Finding f from f' and f''**

A function f is continuous on its domain $[-2, 4]$, $f(-2) = 5$, $f(4) = 1$, and f' and f'' have the following properties.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
f'	$+$	does not exist	$-$	0	$-$
f''	$+$	does not exist	$+$	0	$-$

1. Find where all absolute extrema of f occur.

2. Find where the points of inflection of f occur.

3. Sketch a possible graph of f.